Carmen Bruni

Math 539, Section 201—Homework #1
due Wednesday, January 25, 2012 by 9:59 AM

1. *Show that $\sigma_a - \sigma_c$, the difference between the abscissa of absolute convergence of a Dirichlet series and its abscissa of convergence, can take any value in $[0, 1]$. (Hint: consider linear combinations of $\zeta(s)$ and $\eta(s) = \sum_{n=1}^{\infty}(-1)^{n-1}n^{-s}$, and of their shifts $\zeta(s+c)$ and $\eta(s+c)$.)*

Solution: First, lets note that in class, we saw that $\zeta(s)$ and $\eta(s)$ satisfied the two boundary conditions required by the question. So we just need to show that $\sigma_a - \sigma_c \in (0, 1)$ for all possible values. Let $0 < c < 1$. We examine the Dirichlet series defined by

$$\eta(s) + \zeta(s + c)$$

Let's examine convergence first. From class we know that $\zeta(s + c) = \sum_{n=1}^{\infty} n^{-s-c}$ has an abscissa of convergence equal to $\sigma_c = 1 - c$ (this series diverges at $1 - c$). Also, we know from class that $\eta(s)$ has an abscissa of convergence equal to $\sigma_c = 0$. Since $c \in (0, 1)$, we know that there can't be any cancellation of terms from the power series and so the convergence of the sum occurs whenever these two individual series converge. This is the case when $\sigma > 1 - c$ and so the sum has abscissa of convergence $\sigma_c = 1 - c$. Arguing a bit more formally, we see at $s = 1 - c$ that

$$\sum_{n=1}^{\infty} n^{-c}n^{-1+c} < \sum_{n=1}^{\infty}((-1)^{n+1} + n^{-c})n^{-1+c}$$

which holds either by noting that $\eta(1)$ is positive ($\log(2)$ according to the power series of $\log(1+x)$) which would mean you're adding something that diverges to something thats positive which is still divergent or noting that the first term on the right is one bigger than the first term on the left, the second term on the right is $(-1)2^{-1+c} + 2^{-1}$ and on the left its 2^{-1} so the term on the right is $(-1)2^{-1+c}$ bigger which in absolute value is less than 1, the amount bigger in the first term and then inductively pairwise, we can show this always holds. In the end, the divergence test takes us home (which was very long winded - I'm sorry). To show that the abscissa of convergence is precisely $1 - c$, we need to show that the series converges when $s = 1 - c + \epsilon$ for all $0 < \epsilon < c$. In this case, notice that via a Piazza post (which I'll assume I can use as a valid reference and if not then I guess I know better for next time...)

$$\sum_{n=1}^{\infty}((-1)^{n+1} + n^{-c})n^{-(1-c+\epsilon)} = \sum_{n=1}^{\infty}(-1)^{n+1}n^{-(1-c+\epsilon)} + \sum_{n=1}^{\infty} n^{-c}n^{-(1-c+\epsilon)}$$

The first series converges by the alternating series test and the second series converges by the p-test. Hence our sum above is convergent when $s = 1 - c + \epsilon$. Hence [MV07][p.11] says that $\sigma_c = 1 - c$.

Next, I claim that the abscissa of absolute convergence occurs at $\sigma_a = 1$. Let $\epsilon > 0$, then at $s = 1 + \epsilon$,

$$\sum_{n=1}^{\infty} |(-1)^{n+1} + n^{-c}|n^{-1-\epsilon} \leq \sum_{n=1}^{\infty} n^{-1-\epsilon} + \sum_{n=1}^{\infty} n^{-c-1-\epsilon}$$

and both of these last sums are (absolutely) convergent by the p-test and hence the original series is absolutely convergent for all $\sigma > 1$. To show divergence at $\sigma = 1$, we notice that

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n} < \sum_{n=1}^{\infty}\frac{1}{2n-1} \leq \sum_{n \text{ positive odd}} |(-1)^{n+1} + n^{-c}|\frac{1}{n} \leq \sum_{n} |(-1)^{n+1} + n^{-c}|n^{-1}$$

where the first inequality holds by term-wise comparison, the second inequality holds since whenever n is odd,

$$|(-1)^{n+1} + n^{-c}| = 1 + n^{-c} > 1$$

is true and the last inequality holds because we're adding in more positive terms. The last of the inequalities above is precisely $\eta(1) + \zeta(1+c)$. The divergence test shows us that this value is divergent. Hence, by [MV07][p.11], we have that $\sigma_a = 1$. Hence $\sigma_a - \sigma_c = c \in (0,1)$ and so we are done since c was arbitrary in that interval. ■

2. *Let* $\{q_1, q_2, \dots\} = \{2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, \dots\}$ *denote the sequence of all prime powers. Let* $Q(n)$ *denote the largest prime power dividing* n.

(i) *Prove that* $\lim_{n\to\infty} Q(n) = \infty$.
(ii) *Suppose that* $f(n)$ *is a multiplicative function with the property that* $\lim_{k\to\infty} f(q_k) = 0$. *Must it be true that* $\lim_{n\to\infty} f(n) = 0$ *as well?*

(i) Let $M > 1$. Then, choose $A > 0$ so that $2^A > M$. Now, define $N_0 = \prod_{p \le M} p^A$. Then I claim that $Q(n) > M$ for all $n > N_0$. This holds since we fall in one of two cases. Either n has no prime factors less than M, in which case, it has one bigger than M and so $Q(n) > M$ or we have that n has only prime factors less than M. However, in this case, we know that it must have a prime power strictly bigger than A, in other words, there exists a prime p dividing N so that $v_p(N) > A$. This gives, $Q(n) > p^A \ge 2^a > M$ and so in both cases, we see that $Q(n) > M$. Hence, as M was arbitrary, we have that $Q(n) \to \infty$ as $n \to \infty$ as required. ■

(ii) First notice that since $f(q_k)$ converges as k tends to infinity, we have to have that there exists a N such that $f(q_i) < 1$ for all $i > N$. As in the previous part, choose an N_1 large enough so that $Q(n) > q_k$ for all $n \ge N_1$. Define $C := \prod_{f(q_i)>1}(f(q_i)) + 1$ which is finite by my first remark above. In particular, via the multiplicity of f, we have $f(n/Q(n)) < C$ for all n (in fact we have the stronger $f(n) < C$ but we won't need this). Now, there exists an $N_2 > 0$ such that $f(q_i) < \frac{\epsilon}{C}$ for all $i > N_2$. Choosing $N_3 = \max\{N_1, N_2\}$, we must have that $f(Q(n)) < \frac{\epsilon}{C}$ for all $n > N_3$. Combining this with the fact that unconditionally for all n, we have that $f(n/Q(n)) < C$, we see

$$f(n) = f(Q(n))f(n/Q(n)) < (\epsilon/C)C = \epsilon$$

with the first equality holding since $Q(n)$ was the largest prime power dividing n and so is coprime to $n/Q(n)$ and thus we may use the multiplicity of f. This completes the proof. ■

3. *Let* h *and* k *be functions with* $h(x) > 2$ *and* $k(x) > 2$ *for all sufficiently large* x.

(i) *Prove that* $h(x) \ll k(x)$ *implies* $\log h(x) \ll \log k(x)$.
(ii) *Show via a counterexample that the converse to part (a) is false.*
(iii) *Prove that* $\log h(x) = o\big(\log k(x)\big)$ *implies* $h(x) = o\big(k(x)\big)$.
(iv) *For all real numbers* $A > 0$ *and* $0 < b < 1$ *and* $\epsilon > 0$, *show that*

$$\log^A x \ll \exp(\log^b x) \ll x^\epsilon$$

uniformly for $x \ge 1$. *(The implicit constants may depend upon A, b, and* ϵ*—just not on* x. *By the way, the "uniformly" in* x *is already implied by the definition of* $\ll$, *so it means exactly the same thing to say just that the estimates hold "for* $x \ge 1$*"; but sometimes people add the "uniformly" for emphasis.)*

(i) Let x_0 be the value such that both $h(x)$ and $k(x)$ are greater than 2 for all $x > x_0$. Weird things can happen before x_0, but since we really care about the next three problems in the limit, we will be okay. By definition, there exist a C' such that $h(x) \leq C'k(x)$ for all x sufficiently large (in particular, we at least want $x > x_0$). Choose $C > \max\{C', 1\}$ so that $\log(C) > 0$. Now, notice that in the range of x above, we have that $2 < k(x)$ and so $\log(2) < \log(k(x))$ giving $1 < \frac{\log(k(x))}{\log(2)}$. Hence,

$$\begin{aligned}\log(h(x)) \leq \log(C) + \log(k(x)) &< \log(C)\frac{\log(k(x))}{\log(2)} + \log(k(x)) \\ &= \left(\frac{\log(C)}{\log(2)} + 1\right)\log(k(x))\end{aligned}$$

(ii) Let $h(x) = x^2$ and let $k(x) = x$. Notice that x^2 is not $O(x)$ so $x^2 \not\ll x$ however, $\log(x^2) = 2\log(x)$ and so $\log(h(x)) \ll \log(k(x))$ as required. ■

(iii) By assumption, we have that

$$\lim_{x\to\infty} \frac{\log(h(x))}{\log(k(x))} = 0$$

which implies that for $0 < \epsilon_0 < 1$, we have that there is an M (which can be chosen bigger than x_0) such that $\frac{\log(h(x))}{\log(k(x))} < \epsilon_0$ for all $x > M$. This implies that $h(x) \leq k(x)^{\epsilon_0}$ or equivalently $\frac{h(x)}{k(x)} \leq k(x)^{\epsilon_0 - 1}$ for all $x > M$. Note that the above limit going to 0 either means that $\log(h(x))$ tends to 0 (a contradiction since $h(x) > 2$) or that $\log(k(x))$ tends to infinity in the limit. This means that $k(x)$ tends to infinity and in turn $k(x)^{1-\epsilon_0}$ also tends to infinity. So, for all $\epsilon > 0$, there exists an M_1 (again which can be chosen bigger than x_0) such that $\frac{1}{k(x)^{1-\epsilon_0}} < \epsilon$ for all $x > M_1$. Now, we have that for all $x > \max\{M_1, M\}$

$$\frac{h(x)}{k(x)} \leq k(x)^{\epsilon_0 - 1} = \frac{1}{k(x)^{1-\epsilon_0}} < \epsilon.$$

Hence, this means that

$$\lim_{x\to\infty} \frac{h(x)}{k(x)} = 0$$

and so $h(x) = o(k(x))$ as required. ■

(iv) By the comments in [MV07][p.46-47], we have that it suffices to prove the claim for $x > x_0$ since between 1 and x_0, we have that all the functions in question are bounded away from 0 and from ∞ with the exception of the smallest of the terms $\log^A(x)$ which is okay. In fact, it suffices to show that $\log^A(x) = o(\exp(\log^b(x)))$ and that $\exp(\log^b(x)) = o(x^\epsilon)$ since little-oh is stronger than big-oh. In fact, we can use the previous problem to show that $\log(\log^A(x)) = o(\log^b(x))$ and that $\log^b(x) = o(\log(x^\epsilon))$. So proving the first claim, we have via L'Hopital's rule,

$$\lim_{x\to\infty} \frac{\log(\log^A(x))}{\log^b(x)} = \lim_{x\to\infty} \frac{\frac{1}{\log^A(x)}(A\log^{A-1}(x))\frac{1}{x}}{b\log^{b-1}(x)\frac{1}{x}} = \lim_{x\to\infty} \frac{A}{b\log^b(x)} = 0$$

(so technically the first equality is only validated after the last equality but we're adults now so I think this is okay to write this up like this). This gives us that $\log(\log^A(x)) = o(\log^b(x))$ and hence by the previous problem and previous discussion, we have $\log^A x \ll$

$\exp(\log^b x)$. For the second claim, we have again via L'Hopital's rule,

$$\lim_{x\to\infty} \frac{\log^b(x)}{\log(x^\epsilon)} = \lim_{x\to\infty} \frac{b\log^{b-1}(x)\frac{1}{x}}{\epsilon\frac{1}{x}} = \lim_{x\to\infty} \frac{b}{\epsilon\log^{1-b}(x)} = 0$$

and hence by the previous problem and previous discussion, we have $\exp(\log^b x) \ll x^\epsilon$ completing the proof. ■

4. *Let f be a multiplicative function. We would like to have conditions under which we can conclude that both expressions*

$$\sum_{n=1}^{\infty} f(n) \quad \textit{and} \quad \prod_p \left(1 + f(p) + f(p^2) + \cdots\right) \quad \textit{converge to equal values.} \qquad (*)$$

We know (Theorem 1.9) that assuming $\sum_{n=1}^\infty |f(n)| < \infty$ is one hypothesis that is sufficient to imply $(*)$.

(i) *Prove that assuming*

$$\prod_p \left(1 + |f(p)| + |f(p^2)| + \cdots\right) < \infty$$

is also sufficient to imply $(*)$. (Be rigorous!)

(ii) *Show that assuming*

$$\prod_p \left(1 + |f(p) + f(p^2) + \cdots|\right) < \infty$$

is not *sufficient to imply $(*)$.*

(i) Let $\epsilon' > 0$. Choose $0 < \epsilon < 1$ such that $(1+\epsilon)\epsilon < \epsilon'$. Choose Y so that

$$\prod_{p>Y}\sum_{r=0}^{\infty} |f(p^r)|p^{-\sigma r} < 1+\epsilon.$$

For all primes smaller than or equal to Y (finitely many), pick r_p so that

$$\sum_{r>r_p} |f(p^r)|p^{-\sigma r} < \frac{\epsilon}{2^p}$$

possible since all of these sums must converge as the product of all them converge. Define

$$N(y) = \{n \in \mathbb{Z}^+ : p \mid n \Rightarrow p \le Y, \quad v_p(n) =< r_p\}.$$

The trick is that this $N(Y)$ is chosen so that as Y tends to infinity, $N(Y)$ contains all positive integers and that

$$\prod_{p\le Y}(\sum_{r=0}^{r_p} f(p^r)p^{-s}) = \sum_{n\in N(Y)} f(n)n^{-s}$$

Using this fact, we have that

$$
\begin{aligned}
|\sum_{n\in N(Y)} f(n)n^{-s} - \prod_p \sum_{r=0}^{\infty} f(p^r)p^{-s}| &\leq \left| \prod_{p>Y} (\sum_{r=0}^{\infty} f(p^r)p^{-s}) \prod_{p\leq Y} (\sum_{r>r_p} f(p^r)p^{-sr}) \right| \\
&\leq \prod_{p>Y} (\sum_{r=0}^{\infty} |f(p^r)|p^{-\sigma}) \prod_{p\leq Y} \sum_{r>r_p} |f(p^r)|p^{-\sigma r} \\
&\leq (1+\epsilon) \sum_{p\leq Y} \epsilon/2^p \\
&\leq (1+\epsilon) \sum_{i=1}^{\infty} \epsilon/2^i \\
&\leq (1+\epsilon)\epsilon < \epsilon'
\end{aligned}
$$

Hence, for Y large enough, the modified partial sums will converge to the Euler product. This completes the proof. ■

(ii) I did not mean to look up the answer, but in class you said there were good tidbits in the notes. So I read them and I managed to find the answer to this problem in [MV07][p.32]. I'm sorry about that - it's one of those things too that once you read it its tough to get out of your head so I decided to go ahead and flesh out the details. Define on prime powers

$$
f(1) = 1 \qquad f(p) = 1 \qquad f(p^2) = -1 \qquad f(p^k) = 0 \quad \text{for } k \geq 3
$$

and extend the definition multiplicatively. The product at the point $s = 0$ is

$$
\prod_p (1 + p^{-(0)} - p^{-2(0)} + 0p^{-3(0)} + ...) = \prod_p (1) = 1
$$

However, the $f(n)$ terms themselves do not converge to 0. To see this, notice that $f(p) = 1$ for all primes and $f(p^2) = -1$ for all prime squares. Thus there are two subsequences that tend to different values hence the $f(n)$ do not converge [to 0]. This means that $\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} f(n)n^{-(0)} \neq 1$ which gives an example where the values differ. This contradicts $(*)$ and completes the proof. ■

5. *Let $s(x)$ be any function defined on the interval $[0, 1]$, and define*

$$
F(n) = \sum_{1\leq k\leq n} s\left(\tfrac{k}{n}\right) \qquad \textit{and} \qquad G(n) = \sum_{\substack{1\leq k\leq n \\ (k,n)=1}} s\left(\tfrac{k}{n}\right).
$$

(i) *Prove that $G = F * \mu$.*

(ii) *Evaluate the sum of the primitive nth roots of unity*

$$
\sum_{\substack{1\leq k\leq n \\ (k,n)=1}} e^{2\pi i k/n}
$$

as a function of n.

(i) By the discussion from class, it suffices to show that

$$
G * \mathbf{1} = F
$$

since this would imply $G = f * \mu$. Examining the right hand side of the above, we have that

$$G * \mathbf{1} = \sum_{d|n} G(d)\mathbf{1}(\tfrac{N}{d}) = \sum_{d|n} \sum_{\substack{1\le k\le d \\ (k,d)=1}} s\big(\tfrac{k}{d}\big)$$

Now, I claim that

$$\sum_{d|n} \sum_{\substack{1\le k\le d \\ (k,d)=1}} s\big(\tfrac{k}{d}\big) = \sum_{1\le k\le n} s\big(\tfrac{k}{n}\big) = F(n)$$

which would complete the proof. To show this, it suffices to show that

$$S_n := \{\tfrac{k}{n} : 1 \le k \le n\} = \{\tfrac{k}{d} : d \mid n, \quad 1 \le k \le d, \quad (k,d) = 1\} =: T_n$$

and then this would directly imply that the two finite sums are actually adding the same elements, just labeled differently. So, notice if $\frac{k}{d} \in T_n$, then $d \mid n$ so there exists an integer c such that $cd = n$. Hence

$$\tfrac{k}{d} = \tfrac{kc}{dc} = \tfrac{kc}{n} \in S_n$$

holding since $1 \le k \le d$ and so $1 \le c \le kc \le dc = n$. For the reverse, if $\frac{k}{n} \in S_n$, then let $d := \gcd(k,n)$. This gives us that $\gcd(\frac{k}{d}, \frac{n}{d}) = 1$. Further, as $1 \le k \le n$, we have $1 \le \frac{k}{d} \le \frac{n}{d}$ and clearly $\frac{n}{d} \mid n$. Hence

$$\frac{k}{n} = (\tfrac{k}{d})/(\tfrac{n}{d}) \in T_N.$$

Thus, we have shown that S_n is contained in T_n and that T_n is contained in S_n. This shows the set equality. There is one last small point I think I have to make and that is that the elements of S_n and T_n don't actually repeat. It might be clear form the above proof but I'll just prove this separately just to be on the safe side. For S_n this is dead obvious. For T_n, if $\frac{k}{d} = \frac{j}{e}$, then $k = \frac{dj}{e}$. as $\gcd(k,d) = 1$, we have that $k \mid \frac{j}{e}$ and so $k \mid j$. Similarly, $j \mid k$ and so $j = k$. Given this, its clear that $d = e$ and this completes this mini claim. The proof follows form the preamble discussion. As a corollary of this, notice that

$$n = |S_n| = |T_n| = \sum_{d|n} \phi(d)$$

(not sure if this will be useful but I thought I should note it...) ■

(ii) From the previous part, we know that

$$G(n) = F * \mu(n) = \sum_{d|n} \sum_{1\le k\le d} e^{\frac{2\pi i k}{d}} \mu(\tfrac{n}{d})$$

Let's look at that first sum. If $d = 1$ then the sum is clearly 1. Otherwise, $d > 1$ and this is a geometric series. This is equal to

$$\sum_{1\le k\le d} e^{\frac{2\pi i k}{d}} = \frac{e^{\frac{2\pi i}{d}}((e^{\frac{2\pi i}{d}})^d - 1)}{e^{\frac{2\pi i}{d}} - 1} = \frac{e^{\frac{2\pi i}{d}}(1-1)}{e^{\frac{2\pi i}{d}} - 1} = 0$$

and hence, the above sum reduces to

$$G(n) = \sum_{d|n} \sum_{1\le k\le d} e^{\frac{2\pi i k}{d}} \mu(\tfrac{n}{d}) = \mu(\tfrac{n}{1}) = \mu(n)$$

as required. ■

6. *Prove that the following identities all hold in suitable half-planes (be explicit about which half-planes).*

(i) $\displaystyle\sum_{\substack{n\geq 1\\(n,q)=1}} \frac{1}{n^s} = \zeta(s)\prod_{p|q}\left(1-\frac{1}{p^s}\right)$ *for any positive integer* q

(ii) $\displaystyle\sum_{n=1}^{\infty}\frac{d(n^2)}{n^s} = \frac{\zeta(s)^3}{\zeta(2s)}$

(iii) $\displaystyle\sum_{n=1}^{\infty}\frac{\sigma_a(n)}{n^s} = \zeta(s)\zeta(s-a)$ *for any complex number* a*, where* $\sigma_a(n) = \sum_{d|n} d^a$ *is the generalized sum-of-divisors function.*

(i) Recall that the function on the right converges absolutely whenever $\sigma > 1$. The function on the left will converge absolutely also whenever $\sigma > 1$ since

$$\left|\sum_{\substack{n\geq 1\\(n,q)=1}} \frac{1}{n^s}\right| \leq \sum_{\substack{n\geq 1\\(n,q)=1}} \frac{1}{n^\sigma} \leq \sum_{n=1}^{\infty}\frac{1}{n^\sigma}$$

and the last sum converges [absolutely] when $\sigma > 1$ by the p-test. Define

$$f(n) = \begin{cases} 1 & \text{if } \gcd(n,q) = 1 \\ 0 & \text{otherwise} \end{cases}$$

This function is multiplicative since if m, n are coprime integers, then $\gcd(q, mn) > 1$ implies that at least one of $\gcd(q, m)$ or $\gcd(q, n)$ is greater than 1 and so $f(mn) = 0 = f(m)f(n)$. Further if $\gcd(q, mn) = 1$, then $\gcd(q, mn) = 1$ and $\gcd(q, mn) = 1$ and so $f(mn) = 1 = (1)(1) = f(m)f(n)$. The sum in question is just $\sum_{n=1}^{\infty} f(n)n^{-s}$ and we know from the above that this is absolutely convergent in the half-plane $\sigma > 1$. By [MV07][p. 20], we have

$$\begin{aligned}\sum_{n=1}^{\infty} f(n)n^{-s} &= \prod_{p}(1 + f(p)p^{-s} + f(p^2)p^{-2s} + \ldots)\\ &= \prod_{p\nmid q}(1 + f(p)p^{-s} + f(p^2)p^{-2s} + \ldots)\\ &= \prod_{p\nmid q}(1 + p^{-s} + p^{-2s} + \ldots)\end{aligned}$$

the second last equality holding since $f(p^k) = 0$ if $p \mid q$. Rewriting this, we see

$$\begin{aligned}\sum_{n=1}^{\infty} f(n)n^{-s} &= \prod_{p\nmid q}(1 + p^{-s} + p^{-2s} + \ldots)\\ &= \prod_{p}(1 + p^{-s} + p^{-2s} + \ldots)\prod_{p|q}(1 + p^{-s} + p^{-2s} + \ldots)^{-1}\\ &= \zeta(s)\prod_{p|q}(1 - p^{-s})\end{aligned}$$

where the last equality holds via the Taylor expansion of $(1-x)^{-1}$. ■

(ii) So again the right hand side, the term $\zeta(s)^3$ converges absolutely when $\sigma > 1$. The term $\zeta(2s)$ converges absolutely whenever $\sigma > 0.5$ and so the quotient converges absolutely whenever $\sigma > 1$. The left hand side I believe will also converge for $\sigma > 1$ if I put enough care into the bound, but I will show it for $\sigma > 2$. Basically, the number of divisors of a number n can be no more than $2\sqrt{n}$. We can pair divisors d with $\frac{n}{d}$. At least one of these is less than or equal to $\sqrt{n}$ and hence we have the bound specified. Thus,

$$\left|\sum_{n=1}^{\infty} \frac{d(n^2)}{n^s}\right| \leq \sum_{n=1}^{\infty} \frac{\sqrt{n^2}}{n^\sigma} = \sum_{n=1}^{\infty} \frac{1}{n^{\sigma-1}}$$

and the last sum converges whenever $\sigma > 2$ by the p-test. So our half plane of convergence will be $\sigma > 2$ (though again, we can lower this to $\sigma > 1$ by using the bound $d(n) \ll n^\epsilon$). Now, set $f(n) = d(n^2)$ (which we know is multiplicative). Then for any prime p,

$$f(p^k) = d(p^{2k}) = \sum_{d|p^{2k}} 1 = 2k+1$$

and thus, we have via [MV07][p. 20]

$$\sum_{n=1}^{\infty} \frac{d(n^2)}{n^s} = \prod_p (1 + f(p)p^{-s} + f(p^2)p^{-2s} + ...) = \prod_p (1 + 3p^{-s} + 5p^{-2s} + ...)$$

So it suffices to play around with Taylor series until we get the right match. Notice that

$$\begin{aligned} 1 + 3x + 5x^2 + 7x^3 + 9x^4 + ... &= \sum_{i=0}^{\infty} x^i + (2x + 4x^2 + 6x^3 + 8x^4 + ...) \\ &= \sum_{i=0}^{\infty} x^i + 2x(1 + 2x + 3x^2 + 4x^3 + ...) \end{aligned}$$

Notice that the last sum is precisely the derivative of the power series of $(1-x)^{-1}$ which is equal to $(1-x)^{-2}$ and hence

$$1 + 3x + 5x^2 + 7x^3 + 9x^4 + ... = \frac{1}{1-x} + \frac{2x}{(1-x)^2} = \frac{1+x}{(1-x)^2} = \frac{1-x^2}{(1-x)^3}$$

In our setting, the thing on the left above with $x = p^{-s}$ is one of the terms in our Euler product of our sum. Hence, plugging this in, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{d(n^2)}{n^s} &= \prod_p (1 + 3p^{-s} + 5p^{-2s} + ...) \\ &= \prod_p \frac{1 - p^{-2s}}{(1-p^{-s})^3} \\ &= \frac{\zeta(s)^3}{\zeta(2s)} \end{aligned}$$

remembering that in the last equality, the zeta function has a negative exponent so we have to flip the fraction first. ■

(iii) First, a remark. Since the sigma function defined here is the sum of positive integers to complex powers, my proof form the previous assignment that $\sigma_k(n)$ is multiplicative will carry over. Now, as for half-planes of convergence, the right hand side is absolutely convergent whenever $\sigma > \max\{1, 1+\Re(a)\}$ which again was done in class and in problem 1. The left hand side of the required equality could probably be shown to have the same half plane, but again I'll be a bit sloppy and just show it for some really crude bound of $\sigma > \max\{2, 1+\Re(2a)\}$. From the previous assignment, I showed that if $a \neq 0$, then

$$\sigma_a(p^k) = \frac{p^{(k+1)a} - 1}{p-1}$$

and so we have that

$$\sigma_a(n) = \prod_{p^k \| n} \frac{p^{(k+1)a} - 1}{p-1} = \prod_{p^k \| n} p^{ak} \frac{p^a - 1/p^{ka}}{p-1} = n^a \prod_{p^k \| n} \frac{p^a - 1/p^{ka}}{p-1}$$

Taking absolute values yields

$$|\sigma_a(n)| = n^{\Re(a)} \prod_{p^k \| n} \left| \frac{p^a - 1/p^{ka}}{p-1} \right| \leq n^{\Re(a)} n^{\Re(a)} = n^{2\Re(a)}$$

where the last inequality holds by a really crude upper bound (the fraction above is biggest when k is always 0. This gives a term in the numerator of the large fraction bounded by p^a and taking the product of all the primes dividing n gives the bound n^a). Hence

$$\sum_{n=1}^{\infty} \left| \frac{\sigma_a(n)}{n^s} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^{\sigma - 2\Re(a)}}$$

and the last term converges by the p-test when $\sigma > 1 + 2\Re(a)$. Now, if $a = 0$, then this is the divisor function and the analysis from the previous problem holds to give the absolute convergence when $\sigma > 2$ (again we could lower this to 1 if we wanted). So to be safe, we'll give the half plane of absolute convergence to be $\sigma > \max\{2, 1+2\Re(a)\}$ though I do suspect this could have been reduced to $\max\{1, 1+\Re(a)\}$ with care. To remind us, we have

$$\sigma_0(p^k) = k+1$$

Okay so the half plane has been set up. Let $f(n) = \sigma_a(n)$. When $a = 0$, we have in the half plane of absolute convergence, we have via [MV07][p.20]

$$\sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^s} = \prod_p (1 + 2p^{-s} + 3p^{-2s} + ...) = \prod_p (1 - p^{-s})^{-2} = \zeta(s)^2$$

where the second last equality holds via the previous problem. Now, when $a \neq 0$, we have

$$\sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^s} = \prod_p (1 + f(p)p^{-s} + f(p^2)p^{-2s} + ...) = \prod_p (1 + \left(\frac{p^{2a}-1}{p-1}\right) p^{-s} + \left(\frac{p^{3a}-1}{p-1}\right) p^{-2s} + ...)$$

Now lets examine the other side

$$\zeta(s)\zeta(s-a) = \prod_p (1-p^{-s})(1-p^{-s+a}) = \prod_p (1 - (1+p^a)p^{-s} + p^a p^{-2s})^{-1}$$

So lets try to invert $1 - Ax + Bx^2$ (in our example $A = 1 + p^a$ and $B = p^a$). Let $1 + c_1 x + c_2 x^2 + ...$ be the power series inverse of $1 - Ax + Bx^2$ (this does exist despite the fact that its not instantly immediate - I'll prove it by simply finding it). Now,

$$\begin{aligned}1 &= (1 - Ax + Bx^2)(1 + c_1x + c_2x^2 + c_3x^3 + ...)\\ &= 1 + (c_1 - A)x + (B - Ac_1 + c_2)x^2 + (Bc_1 - Ac_2 + c_3)x^3 + ...\end{aligned}$$

and the general term is $(Bc_{n-2} - Ac_{n-1} + c_n)x^n$. each of the coefficients above are 0 and so, we have $c_1 = A = 1 + p^a = \frac{p^{2a}-1}{p-1}$. Then $B - Ac_1 + c_2 = 0$ and so

$$c_2 = Ac_1 - B = (1 + p^a)^2 - p^a = 1 + p^a + p^{2a} = \frac{p^{3a} - 1}{p - 1}$$

and so lets go inductively and assume that $c_m = \frac{p^{(m+1)a}-1}{p-1}$ for all $m < n$. Then for c_n,

$$\begin{aligned}c_n = Ac_{n-1} - Bc_{n-2} &= (1 + p^a)\frac{p^{na} - 1}{p - 1} - p^a\frac{p^{(n-1)a} - 1}{p - 1}\\ &= \frac{p^{na} - 1 + p^{(n+1)a} - p^a - p^{na} + p^a}{p - 1}\\ &= \frac{p^{(n+1)a} - 1}{p - 1}\end{aligned}$$

which completes the induction proof. Hence, combining this, we have

$$\begin{aligned}\zeta(s)\zeta(s-a) &= \prod_p (1 - (1 + p^a)p^{-s} + p^a p^{-2s})^{-1}\\ &= \prod_p (1 + \left(\frac{p^{2a} - 1}{p - 1}\right)p^{-s} + \left(\frac{p^{3a} - 1}{p - 1}\right)p^{-2s} + ...)\\ &= \sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^s}\end{aligned}$$

which completes the proof. ■

7. *Define the Dirichlet series* $P(s) = \sum_p \frac{1}{p^s}$ *and* $W(s) = \sum_{n=1}^{\infty} \frac{\omega(n)}{n^s}$*, where* $\omega(n)$ *is the number of distinct prime factors of* n*.*

(i) *What is the abscissa of convergence for* $P(s)$*?*
(ii) *Prove that formally,* $W(s) = \zeta(s)P(s)$*.*
(iii) *What is the abscissa of convergence for* $W(s)$*?*
(iv) *Prove that* $\sum_{n \le x} \omega(n) \ll_\epsilon x^{1+\epsilon}$ *for every* $\epsilon > 0$*.*
(v) *Can* $W(s)$ *be analytically continued to an entire function?*

(i) As all terms of $P(s)$ are non-negative, the abscissa of convergence equals the abscissa of absolute convergence (as was discussed in class and in [MV07][p. 11]). Notice that

$$|P(s)| \le \sum_p \frac{1}{p^\sigma}$$

and the last sum is divergent when $\sigma = 1$ by a proof of Erdos in the 1920s (or by Euler but I've always had a love hate relationship with Erdos' proof) and is convergent when $\sigma > 1$ by the p-test (I guess we'd have to add in missing terms to actually use the p-test but thats okay it won't affect convergence in this situation). Hence the abscissa of convergence is $\sigma_c = \sigma_a = 1$. ■

(ii) So I'll assume that by formally, we mean symbolically and then later we'll justify convergence so let's prove this. Let

$$f(n) = \begin{cases} 1 & \text{if } n = p \\ 0 & \text{otherwise} \end{cases}$$

Then, $P(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$. Hence formally by [MV07][p. 19], we have

$$\begin{aligned} \zeta(s)P(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \sum_{n=1}^{\infty} \left(\sum_{k+m=n} f(k)\mathbf{1}(m) \right) \frac{1}{n^s} \\ &= \sum_{n=1}^{\infty} \left(\sum_{p \leq n} 1 \right) \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{\omega(n)}{n^s} = W(s) \end{aligned}$$

completing the proof. ■

(iii) The previous problem showed that $W(s) = \zeta(s)P(s)$. Applying [MV07][p. 19], we can see that the abscissa of absolute convergence of $W(s)$ is at best 1, that is, $\sigma_a \leq 1$. To show that its exactly 1, we note that $P(1) \leq W(1)$ and that $P(1)$ diverges (so it probably is being sloppy to write $P(1)$ since its not a value but the idea is that the divergent series is less than $W(1)$ term-wise). The inequality holds since $1 = w(p)$ for every prime p. Hence, the abscissa of absolute convergence is $\sigma_a = 1$. As the series $W(n)$ is positive, we have that the abscissa of absolute convergence is equal to the abscissa of convergence which completes the proof. ■

(iv) Let $A(x) = \sum_{n \leq x} \omega(n)$. From class and using [MV07][p. 13], we have that

$$\limsup_{x \to \infty} \frac{\log |A(x)|}{\log(x)} = \sigma_c = 1$$

and this is equivalent to $A(x) \ll_\epsilon x^{\sigma_c + \epsilon}$. ■

(v) It is not possible for $W(s)$ to be analytically continued to an entire function. Notice that $W(s)$ has all non-ngeative terms and a finite abscissa of convergence. Hence it satisfies Landau's Theorem [MV07][p.16] and so σ_c is a singularity of $W(s)$. This completes the proof (though does not answer the question can it be meromorphically continued?) ■

8.

(i) Show that if $\alpha(s) = \sum a_n n^{-s}$ has abscissa of convergence $\sigma_c < \infty$, then

$$\lim_{\sigma \to \infty} \alpha(\sigma) = a_1$$

(ii) Show that $\zeta'(s) = -\sum_{n=1}^{\infty} \log(n) n^{-s}$ for $\sigma > 1$.

(iii) Show that

$$\lim_{\sigma \to \infty} \zeta'(\sigma) = 0$$

(iv) Show that there is no half plane in which $1/\zeta'(s)$ can be written as a convergent Dirichlet series.

(i) Really what we want is to take the limit pointwise. To accomplish this, we appeal to the Lebesgue dominated convergence theorem. Our Dirichlet series in question has finite abscissa of convergence. So noting that $\sigma_a \leq \sigma_c + 1$, we have for any fixed $\sigma' > \sigma_c + 2$ (to be safe with boundary contidions) that the sequence $\sum |a_n| n^{-\sigma'} < \infty$ and dominates our sequence when we choose $\sigma > \sigma'$. We have the pointwise convergence requirement and thus we can appeal to LDCT to see that

$$\lim_{\sigma\to\infty} \alpha(\sigma) = \sum_{n=1}^{\infty} \lim_{\sigma\to\infty} a_n n^{-\sigma} = a_1 + \sum_{n=2}^{\infty} \lim_{\sigma\to\infty} a_n n^{-\sigma} = a_1$$

which completes the proof. ■

(ii) Here, we want to take the derivative pointwise. To do this we appeal to [MV07][p. 14] which tells us that a Dirichlet series is uniformly bounded in the half-plane $\sigma > \sigma_a + \epsilon$ where $\epsilon > 0$ is arbitrary. So in this region, not only are we absolutely convergent, but also uniformly bounded and so for $\sigma > 1$, we have

$$\zeta'(s) = \frac{d}{ds}\sum_{n=1}^{\infty} n^{-s} = \sum_{n=1}^{\infty} \frac{d}{ds} n^{-s} = -\sum_{n=1}^{\infty} \log(n) n^{-s}$$

as required. ■

(iii) If we can show that $\zeta'(s)$ is absolutely convergent, then we can use the previous two parts, to see that

$$\lim_{\sigma\to\infty} \zeta'(\sigma) = a_1 = \log(1) = 0$$

which would complete the proof. Absolute convergence follows since we have that for all $\epsilon > 0$, $\log(n) \ll n^{\epsilon}$ and so there exists a $C > 0$ such that $\log(n) <= Cn^{\epsilon}$ for all n. Thus

$$|\zeta'(s)| \leq \sum_{n=1}^{\infty} |\log(n)| n^{-\sigma} \leq \sum_{n=1}^{\infty} n^{-\sigma+\epsilon}$$

and the last sum converges whenever $\sigma > 1 + \epsilon$. Since ϵ was arbitrary, this series has abscissa of absolute convergence at most 1 and hence is finite (this sequence DOES converge if say $\sigma = 2$). This shows convergence and justifies the steps above. ■

(iv) Assume towards a contradiction that there was a Dirichlet series, say $\beta(s) = \sum b_n n^{-s}$ that had an abscissa of convergence $\sigma_c < \infty$ so that $1/\zeta'(s) = \beta(s)$. Then $1 = \zeta'(s)\beta(s)$. In particular, this would have to hold for any σ suitably large so that both sequences absolutely converge. Hence, we have that $1 = \zeta'(\sigma)\beta(\sigma)$ for all suitably large σ. Taking the limits of both sides and using all the previous parts, we have

$$1 = \lim_{\sigma\to\infty} (\zeta'(\sigma)\beta(\sigma)) = \lim_{\sigma\to\infty} \zeta'(\sigma) \lim_{\sigma\to\infty} \beta(\sigma) = (0)(b_1) = 0$$

which is a contradiction. ■

9.

(i) *Show that for $\sigma > 0$, we have*

$$\sum_{n=1}^{\infty} (-1)^{n-1} n^{-s} = (1 - 2^{1-s})\zeta(s)$$

(ii) *Show that*

$$\sum_{n=1}^{\infty} (-1)^n \log(n) n^{-1} = C_0 \log(2) - \frac{1}{2}(\log(2))^2$$

where C_0 is Euler's constant.

(iii) *Show that $\sum_{n=1}^{\infty} (-1)^n n^{-1+2012\pi i/\log 2} = 0$. (In analytic number theory,* $\log$ *always denotes the natural logarithm.)*

(i) First lets show this in the region of absolute convergence, which we know is $\sigma > 1$. Notice that in this half-plane,

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s} &= \sum_{n=1}^{\infty}(-1)^{n-1}n^{-s} + 2\sum_{n=1}^{\infty}(2n)^{-s} - 2\sum_{n=1}^{\infty}(2n)^{-s} \\ &= \sum_{n=1}^{\infty} n^{-s} - 2^{1-s}\sum_{n=1}^{\infty}(n)^{-s} \\ &= (1-2^{1-s})\zeta(s).\end{aligned}$$

Now for some analytic trickery. Notice that $1-2^{1-s}$ is entire. Further, by [MV07][p.25], we have that $\zeta(s)$ can be analytically continued to $\sigma > 0$ with the exception of the simple pole at $s = 1$. Hence, the function $(1-2^{1-s})\zeta(s)$ has an analytic continuation to the half plane $\sigma > 0$ since the pole of $\zeta(s)$ at $s = 1$ is eliminated by the zero of $1-2^{1-s}$ at $s = 1$. The analytic continuation of $\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s}$ to $\sigma > 0$ is indeed itself (as its convergent on this region) and hence by the uniqueness of analytic continuation (or by the identity theorem), we have that the equality above holds when $\sigma > 0$. ■

(ii) First, we want to note that the right hand side of the equality in the previous problem has a Taylor series around $s = 1$, namely by taking the Laurent series for $\zeta(s)$ given by

$$\zeta(s) = \frac{1}{s-1} + C_0 + \sum_{n=1}^{\infty} a_n(s-1)^n$$

and the Taylor series of $1-2^{1-s}$ given by

$$1-2^{1-s} = 1-e^{(1-s)\log(2)} = 1-1-\sum_{n=1}^{\infty}\frac{((1-s)\log(2))^n}{n!} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(s-1)^n\log^n(2)}{n!}$$

we get that

$$\begin{aligned}(1-2^{1-s})\zeta(s) &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(s-1)^n\log^n(2)}{n!}\left(\frac{1}{s-1} + C_0 + \sum_{n=1}^{\infty}a_n(s-1)^n\right) \\ &= \log(2) + (C_0\log(2) - \frac{\log(2)^2}{2})(s-1) + \sum_{n=2}^{\infty} b_n(s-1)^n\end{aligned}$$

for some sequence b_n. As the above function is analytic, we have that we can take the derivative to get

$$\frac{d}{ds}(1-2^{1-s})\zeta(s) = C_0\log(2) - \frac{\log(2)^2}{2} + \sum_{n=2}^{\infty} nb_n(s-1)^{n-1}$$

Plugging in $s = 1$ gives us $C_0\log(2) - \frac{\log(2)^2}{2}$. Now this by the previous part must equal the derivative of

$$\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s}$$

with respect to s at the point $s = 1$. By [MV07][p.12], the series above is locally uniformly convergent for $\sigma > \sigma_c = 0$. Hence, we can take the derivative term wise. Further, as the function is analytic there, its derivative is analytic and hence we can simply plug in the value at $s = 1$. This gives (when we combine it with the previous discussion)

$$C_0 \log(2) - \frac{\log(2)^2}{2} = \frac{d}{ds}\bigg|_{s=1} \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s}$$
$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{d}{ds}\bigg|_{s=1} n^{-s} = \sum_{n=1}^{\infty} (-1)^n \log(n) n^{-1}$$

and this completes the proof. ■

(iii) Notice that as $s = 1 - \frac{2012\pi i}{\log(2)}$, we can use the first part of the previous problem to see that

$$\sum_{n=1}^{\infty} (-1)^n n^{-1+2012\pi i/\log 2} = (1 - 2^{1-(1-\frac{2012\pi i}{\log(2)})})\zeta(1 - \tfrac{2012\pi i}{\log(2)})$$
$$= (1 - 2^{-\frac{2012\pi i}{\log(2)}})\zeta(1 - \tfrac{2012\pi i}{\log(2)})$$
$$= (1 - e^{-\log(2)\frac{2012\pi i}{\log(2)}})\zeta(1 - \tfrac{2012\pi i}{\log(2)})$$
$$= (1 - \frac{1}{e^{2012\pi i}})\zeta(1 - \tfrac{2012\pi i}{\log(2)})$$
$$= (1 - 1)\zeta(1 - \tfrac{2012\pi i}{\log(2)}) = 0$$

10. *The* generalized divisor function $d_k(n)$ *is defined, for any positive integer k, to be the number of ordered k-tuples $(m_1, \dots, m_k)$ of positive integers such that $m_1 \times \cdots \times m_k = n$, so that $d_2(n) = d(n)$, for example.*

(i) *Prove that $d_j * d_k = d_{j+k}$ for all positive integers j and k. Given this relationship, what do you think a sensible way to define $d_{1/2}$ would be? Calculate $d_{1/2}(539)$ and $d_{1/2}(16)$. Can you write down a formula for $d_{1/2}(p^r)$ as a function of r?*

(ii) *Prove that $\sum_{n=1}^{\infty} d_k(n) n^{-s} = \zeta(s)^k$, in a suitable half-plane, for all positive integers k. Given this relationship, what do you think a sensible way to define d_z would be for any complex number z? Calculate $d_i(539)$ and $d_i(16)$ (where $i = \sqrt{-1}$).*

(i) We prove the first claim by induction. Our Induction statement will be "For all j, we have that $d_j * d_k = d_{j+k}$". First, for $k = 1$, we have that for any $n \in \mathbb{Z}^+$,

$$d_j * d_1(n) = \sum_{d|n} d_j(d) d_1(\tfrac{n}{d}) = \sum_{d|n} d_j(d) = d_{j+1}(n)$$

where the second to last equality holds since $d_1(m) = 1$ for all m and the last equality holds since in order to write $d_{j+1}(n)$, we can argue that first we have to choose the first j terms. Upon doing that, we notice that the first j terms must multiply together to be a factor of n call it d. The last term must be $\frac{n}{d}$. So all we need to do is count up the number of ways the first j terms can be any of the factors of n, which is precisely the sum in the second to last equality above. As n was arbitrary, the claim is true for $k = 1$. Suppose the claim is true for all integers less than k. Then notice that in particular, the above shows us that

$$d_{k-1} * d_1 = d_k.$$

So now, we have

$$d_j * d_k = d_j * (d_{k-1} * d_1) = (d_j * d_{k-1}) * d_1 = d_{j+k-1} * d_1 = d_{j+k}$$

where we used the associativity of the Dirichlet convolution, and then the induction hypothesis in the last two steps (once with j and $k-1$ and once with $j+k-1$ and 1). This completes the induction proof and the first part of this problem. I'm going to omit the proof that d_k is multiplicative because its not necessary to the argument other than the first thing I'm about to say. As d_k is multiplicative, it makes sense to define $d_{1/2}$ to be the multiplicative function so that $d_{1/2} * d_{1/2} = d_1 = 1$. Let's examine the left hand side on prime powers p^l.

$$1 = (d_{1/2} * d_{1/2})(p^l) = \sum_{d|p^l} d_{1/2}(d) d_{1/2}(\tfrac{p^l}{d}) = \sum_{j=0}^{l} d_{1/2}(p^j) d_{1/2}(p^{l-j}).$$

This last term looks pretty close to a term that could appear in a power series. In fact, the initial comments in [MV07][p.39] led me to considering the power series defined by

$$A(x) := \sum_{l=0}^{\infty} d_{1/2}(p^l) x^l$$

Notice that

$$A(x)^2 = \sum_{l=0}^{\infty} \left(\sum_{j=0}^{l} d_{1/2}(p^j) d_{1/2}(p^{l-j}) \right) x^l = \sum_{l=0}^{\infty} x^l = \frac{1}{1-x}$$

giving us that $A(x) = \pm(1-x)^{-0.5}$. Using the binomial series expansion, we have

$$(1-x)^{-0.5} = \sum_{j=0}^{\infty} \binom{\frac{-1}{2}}{j} (-x)^j$$

where the binomial term is the generalized binomial coefficient. Simplifying, we have

$$\begin{aligned}
\binom{\frac{-1}{2}}{j} &= \frac{\frac{-1}{2}(\frac{-1}{2}-1)...(\frac{-1}{2}-j+1)}{j!} \\
&= \frac{(1)(3)(5)...(2j-1)}{(-2)^j j!} \\
&= \frac{(1)(3)(5)...(2j-1)2^j j!}{(-1)^j((2)^j j!)^2} \\
&= \frac{(2j)!}{(-4)^j j! j!} = \binom{2j}{j} \frac{1}{(-4)^j}.
\end{aligned}$$

where we note $2^j j! = (2)(4)(6)...(2j)$. Plugging into $A(x)$ yields

$$A(x) = \pm(1-x)^{-0.5} = \sum_{j=0}^{\infty} \binom{\frac{-1}{2}}{j} (-x)^j = \sum_{j=0}^{\infty} \binom{2j}{j} \frac{1}{4^j} x^j$$

Since we have determined that $d_{1/2}$ should be multiplicative, we actually want to take the positive square root for $A(x)$ and hence, we have that $d_{1/2}(p^j) = \binom{2j}{j}\frac{1}{4^j}$. As $539 = 7^2 \cdot 11$ and $16 = 2^4$, we have

$$d_{1/2}(539) = d_{1/2}(7^2) d_{1/2}(11) = \left(\binom{2(2)}{2} \frac{1}{4^2} \right) \left(\binom{2}{1} \frac{1}{4} \right) = \frac{6}{16} \cdot \frac{1}{2} = \frac{3}{16}$$

and that

$$d_{1/2}(16) = d_{1/2}(2^4) = \binom{2(4)}{4} \frac{1}{4^4} = \frac{70}{256} = \frac{35}{128}.$$

This concluses the proof. ■

(ii) Throughout for simplicity, I will work with the half plane of absolute convergence for $\zeta(s)$ (so $\sigma > 1$). We show that $\sum_{n=1}^{\infty} d_k(n)n^{-s} = \zeta(s)^k$ by induction. For $k = 1$, the claim is clear since $d_1(n) = 1$ for all positive integers n. Now, suppose the claim is true for $k-1$. Then using [MV07][p.19], we have

$$\begin{aligned}
\zeta(s)^k = \zeta(s)^{k-1}\zeta(s) &= \sum_{n=1}^{\infty} d_{k-1}(n)n^{-s} \sum_{n=1}^{\infty} d_1(n)n^{-s} \\
&= \sum_{n=1}^{\infty} \left(\sum_{ab=n} d_{k-1}(a)d_1(b) \right) n^{-s} = \sum_{n=1}^{\infty} \left(\sum_{a|n} d_{k-1}(a)d_1(\tfrac{n}{a}) \right) n^{-s} \\
&= \sum_{n=1}^{\infty} (d_{k-1} * d_1)(n)n^{-s} = \sum_{n=1}^{\infty} d_k(n)n^{-s}
\end{aligned}$$

where the last equality holds by the previous problem. This completes the induction and hence the proof. To extend this to complex z, first we need to worry about what $\zeta(s)^z$ means. I would like to say $\zeta(s)^z = e^{z\log(\zeta(s))}$ but there are some concerns that need to be tended to. Firstly, notice that because of the convergence of the Euler product in the half plane $\sigma > 1$, we have that $\zeta(s) \neq 0$ in this half plane. Hence this definition can make sense. However, we have to worry about branch cutting as well. The most natural branch cut to take in this case is the one that is real valued on the real axis (in particular for $\sigma > 1$). This is in part to help us match up our definition with the previous part. This should take of the analytical concerns. As before, we would like $d_z(n)$ to be multiplicative and to satisfy the identity above. Looking at the Euler product expansion of $\zeta(s)^z$, we have

$$\sum_{n=1}^{\infty} d_z(n)n^{-s} = \zeta(s)^z = \prod_p (1-p^{-s})^{-z}$$

Now, when we look at the terms for each prime expanded as a power series, we can get values of $d_z(p^j)$. Let's evaluate the last term using the Binomial series expansion. First, notice that

$$\begin{aligned}
\binom{-z}{j} &= \frac{-z(-z-1)...(-z-j+1)}{j!} \\
&= \frac{(-1)^j(z+j-1)(z+j-2)...(z+1)(z)}{j!} = (-1)^j\binom{z+j-1}{j}
\end{aligned}$$

and so

$$\begin{aligned}
(1-p^{-s})^{-z} &= \sum_{j=0}^{\infty} \binom{-z}{j}(-p^{-s})^j = \sum_{j=0}^{\infty} (-1)^j\binom{z+j-1}{j}(-p^{-s})^j \\
&= \sum_{j=0}^{\infty} \binom{z+j-1}{j} p^{-sj}.
\end{aligned}$$

So, a sensible definition of $d_z(p^j)$ would be $d_z(p^j) := \binom{z+j-1}{j}$ which conveniently enough matches up with the previous definition from the last problem. Using similar methods as before, we have

$$d_i(539) = d_i(7^2)d_i(11) = \binom{i+2-1}{2}\binom{i+1-1}{1} = \frac{(i+1)(i)}{2} \cdot i = \frac{-1-i}{2}$$

and that

$$d_i(16) = d_i(2^4) = \binom{i+4-1}{4} = \frac{(i+3)(i+2)(i+1)i}{4!} = \frac{(-1+3i)(1+3i)}{24} = \frac{-5}{12}.$$

This completes the assignment! ■

Acknowledgements: Thanks to Asif for his collaboration on this assignment and a mention to Chenlong for discussing problems 2 and 3 with me.

REFERENCES

[MV07] Hugh L. Montgomery and Robert C. Vaughan. *Multiplicative number theory. I. Classical theory*, volume 97 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2007.